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DEFORMATION OF AN ELASTIC WEDGE REINFORCED BY A BEAM

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Two problems of plane strain of an elastic infinite wedge reinforced by an infinite constant-thickness beam are considered. In the first problem the beam is welded to the wedge along the bisectrix and is in complete contact with it. A longitudinal force, a transverse force, and a bending moment are applied to the end of the beam and arbitrary normal and tangential stresses are given on the boundary surfaces of the wedge. In the second problem, the beam is in contact without friction with one face of the wedge, arbitrary stress resultants act on both the wedge and the beam. Both problems are reduced to first-order difference equations and are solved in closed form.

1. In an elastic wedge let $0 \leq r < \infty$, $-\alpha \leq \theta \leq \alpha$, an elastic beam $2h$ thick (Fig. 1) is welded along the $\theta = 0$ axis, and the contact surfaces of the wedge and beam are connected completely. A longitudinal tensile force $2T$, a bending moment $2M$, a transverse force $2P$ or another load causing equivalent stress resultants at the point $r=0$ of the beam act on the free part of the beam $\theta = \pi$. Concentrated forces, a normal $2N$ and tangential $2S$, are applied to the wedge at its face $\theta = \alpha$ as an arbitrary load.

Let us divide the given load into a symmetric and antisymmetric load. Then because of the symmetry of the elastic domain, the solution of the problem formulated can be

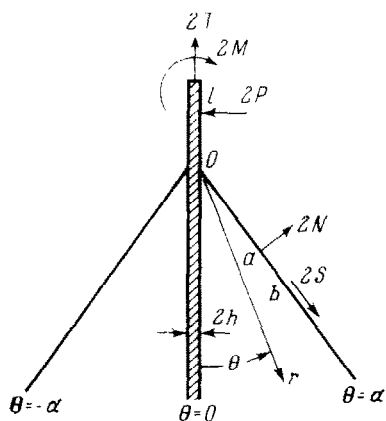


Fig. 1

represented as the sum of solutions: (a) of the problem for half the wedge $0 \leq \theta \leq \alpha$ reinforced by a rod of thickness h under the conditions

$$v(r, 0) = 0, \quad Eh \frac{\partial^2 u(r, 0)}{\partial r^2} + \quad (1.1)$$

$$\tau_{r\theta}(r, 0) = 0, \quad \int_0^\infty \tau_{r\theta}(r, 0) dr = T$$

$$\sigma_\theta(r, \alpha) = N\delta(r - a), \quad (1.2)$$

$$\tau_{r\theta}(r, \alpha) = S\delta(r - b)$$

and (b) the problem for the same wedge under the conditions (1.2) and

$$u(r, 0) = 0, \quad \int_0^\infty \sigma_\theta(r, 0) dr = P, \quad \int_0^\infty \sigma_\theta(r, 0) r dr = M - Pl \quad (1.3)$$

$$D \frac{\partial^4 v(r, 0)}{\partial r^4} - \sigma_\theta(r, 0) = 0, \quad D = \frac{Eh^3}{3(1-\mu^2)} \quad (1.4)$$

Here E and μ are the elastic modulus and Poisson's ratio of the beam and $\delta(r)$ is the Dirac delta function.

Taking into account the general condition (1.2), we shall seek the solution of both problems in the form of Mellin integrals [1]

$$2G \begin{Bmatrix} u(r, \theta) \\ v(r, \theta) \end{Bmatrix} = \frac{1}{2\pi i} \int_L \left(\pm A(p) \left[(p \pm \kappa) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (p+1)\theta \pm \right. \right. \quad (1.5)$$

$$\left. \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (p-1)\theta - \Delta_1^+ \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (p-1)\theta \right] + B(p) \left[(p \pm \kappa) \times \right.$$

$$\left. \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (p+1)\theta + \Delta_1^- \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (p-1)\theta \pm \Delta_2^+ \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (p-1)\theta \right] \pm$$

$$\frac{Na^p}{p} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} |(p-1)(\alpha-\theta)| \mp \frac{Sb^p}{p} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} |(p-1)(\alpha+\theta)| \Big) \frac{dr}{r^p}$$

$$\Delta_1^\pm = \cos 2p\alpha \pm p \cos 2\alpha, \quad \Delta_2^\pm = \sin 2p\alpha \pm p \sin 2\alpha$$

where $\kappa = 3-4\nu$, G and ν are the shear modulus and Poisson's ratio of the wedge, respectively. In order for displacements at the point $r = 0$ to be bounded, we take the line $\text{Re } p = \lambda, -1/4 < \lambda < 0$ as L .

Let us consider problem (a). It follows from the first condition (1.1) and from (1.5) that

$$B(p) = (\Delta_2^+)^{-1} [A(p) (\Delta_1^+ - p + \kappa) - Na^p p^{-1} \sin (p-1)\alpha + \quad (1.6)$$

$$Sb^p p^{-1} \cos (p-1)\alpha]$$

Substituting (1.5), (1.6) into the second condition of (1.1), we obtain

$$\frac{1}{2\pi i} \int_L (p+1) [F(p)A(p) + f(p)] \frac{dp}{r^{p+2}} - \frac{1}{2\pi i} \int_{L_1} (p+1) \times \tag{1.7}$$

$$A(p+1) \frac{dp}{r^{p+2}} = 0$$

$$F(p) = \frac{Ehp\Delta_3}{2G(1+\kappa)\Delta_2^+}, \quad \Delta_3 = 4\kappa \sin^2 p\alpha + 4p^2 \sin^2 \alpha - (1+\kappa)^2$$

$$f(p) = \frac{Eh}{2G(1+\kappa)} \left\{ \frac{\Delta_1^- + p + \kappa}{\Delta_2^+} [Na^p \sin(p-1)\alpha -$$

$$Sb^p \cos(p-1)\alpha] - Na^p \cos(p-1)\alpha + Sb^p \sin(p-1)\alpha \right\}$$

where L_1 is the line $\text{Re } p = \lambda - 1$. We assume that the function $pA(p)$ is (1) regular and (2) tends to zero in the strip $\lambda \leq \text{Re } p \leq \lambda + 1$ as $|\text{Im } p| \rightarrow \infty$. Then by the Cauchy theorem, without altering the integrand, L can be written in (1.7) instead of L_1 and it can be satisfied by solving the first-order difference equation

$$A(p+1) = F(p)A(p) + f(p) \tag{1.8}$$

Similar boundary value problems in polar coordinates with boundary conditions of the third kind have been reduced to difference equations in [2-5], and others.

According to the general solution of (1.8) [6], for $S = N = 0$, i. e. for $f(p) = 0$

$$A(p) = \frac{C}{\sin \pi p} \left[\frac{Eh(1+\kappa)}{2G(\sin 2\alpha + 2\alpha)} \right]^p \prod_{k=1}^{\infty} \frac{\Gamma(a_k + p) \Gamma(1 + b_k - p) b_k^{2p-1}}{\Gamma(b_k + p) \Gamma(1 + a_k - p) a_k^{2p-1}} \tag{1.9}$$

$$C = 1/2\pi (1 + \kappa)^{-1} T$$

Here $\Gamma(p)$ is the gamma function, a_k and b_k are the zeros and poles of the function $F(p)$ which are in the half-plane $\text{Re } p > 0$, the constant C is determined from the third condition in (1.1) whose left side is the Mellin transform of the function $\tau_{r,0}(r, 0)$ equal to $(1 + \kappa)pA(p)$. An arbitrary periodic function, $\sin^{-1}\pi p$ in this case, is selected from conditions (1) and (2). Using the Stirling formula, the absolute convergence and algebraic growth of the infinite product (1.9) in the strip $\lambda \leq \text{Re } p \leq \lambda + 1$ can be given a foundation. Therefore, conditions (1) and (2) are satisfied, $|A(\lambda + i\beta)| = O(|\beta|^{-\epsilon} e^{-\pi|\beta|})$, the integrals (1.5) converge absolutely and uniformly in the whole elastic domain up to its left boundary.

The stresses $\tau_{r,0}(r, 0)$ at the angular point $r = 0$ of the wedge are characterized by the residue at the first pole of the function $A(p)$ to the left of L . For $\alpha < \alpha^*$ they are finite, for $\alpha = \alpha^*$ they have a logarithmic, and for $\alpha > \alpha^*$ a power-law $r^{\alpha-1}$ singularity. Tables of the values of α_1 and the formula $\alpha^* = \arcsin \sqrt{1 - \nu}$ are presented on page 149 in the monograph [1].

Recently, a new canonical solution of remarkable simplicity and efficiency has been found in [5] for the homogeneous equation (1.8). In this case, with some supplementation it acquires the following form:

$$A_0(p) = Q^p \cos^{-1}(1/2\pi p) \Gamma(p) X(p), \quad Q = Eh\kappa[G(1 + \kappa)]^{-1} \quad (1.10)$$

$$X(p) = K^{-1}(p)Y(p) \quad (-1 < \operatorname{Re} p \leq 0), \quad X(p) = Y(p) \quad (0 < \operatorname{Re} p \leq 1)$$

$$Y(p) = \exp \left\{ \frac{1}{2i} \int_{-i\infty}^{i\infty} \operatorname{ctg} \pi(t-p) \ln K(t) dt \right\}$$

$$K(p) = - \frac{F(p)}{Qp} \operatorname{tg} \frac{\pi p}{2}$$

In contrast to (1.9), firstly it is not necessary to evaluate the poles and zeros of the function $F(p)$ here, and secondly, the integral (1.10) has exponential convergence and the infinite product (1.9) is just a power-law. However, the complete solution in [5] is awkward and expressed in triple rather than double integrals because of the reduction of the homogeneous problem of elasticity theory to an inhomogeneous difference equation (the force is applied only to the end of the rod in [5], i. e. $N = S = 0$). Hence, the method of (1.5)–(1.8), (1.10), which is also applicable to three-dimensional problems, clearly yields the most economical solution.

Let us turn to the inhomogeneous equation (1.8) and write its solution in the form

$$A(p) = A_0(p)[C + \cos \pi p Z(p)] \quad (1.11)$$

Substituting (1.11) into (1.8), we obtain the difference equation

$$Z(p+1) = -Z(p) - g(p) \quad (1.12)$$

which has the solution

$$Z(p) = W(p) - g(p) \quad (-1 < \operatorname{Re} p \leq 0), \quad Z(p) = W(p) \quad (0 < \operatorname{Re} p \leq 1) \quad (1.13)$$

$$W(p) = \frac{1}{2i} \int_{-i\infty}^{i\infty} \frac{g(t) dt}{\sin(t-p)\pi}$$

by virtue of the Sokhotskii–Plemelj formulas.

From the third condition in (1.1)

$$\lim_{p \rightarrow 0} \{(1 + \kappa) p A_0(p)[C + \cos \pi p Z(p)]\} = T$$

and from (1.10), we obtain

$$C = T(1 + \kappa)^{-1} X^{-1}(0) - Z(0) \quad (1.14)$$

The validity of conditions (1) and (2) follows from the properties of Cauchy-type integrals and from the estimate $|f(\lambda_1 + i\beta)| = O(|\beta|e^{-\alpha|\beta|})$ for $\lambda \leq \lambda_1 \leq \lambda + 1$.

The inhomogeneous equation is solved by another method in [5].

Let us examine problem (b). By virtue of the first condition in (1.3)

$$A(p) = -(\Delta_2^-)^{-1} [B(p)(\Delta_1^- + p + \kappa) + Na^p p^{-1} \cos(p-1)\alpha - Sb^p p^{-1} \sin(p-1)\alpha]$$

By analogy with (1.7), condition (1.4) can be written as

$$\frac{1}{2\pi i} \int_L (p+3) [F_1(p) F_2(p) B(p) + f(p)] \frac{dp}{r^{p+4}} - \frac{1}{2\pi i} \int_{L_1} (p+3) B(p+3) \frac{dp}{r^{p+4}} = 0 \tag{1.15}$$

$$F_1(p) = Qp(p+1)(p+2), \quad F_2(p) = -\Delta_3(2\kappa\Delta_2^{-1})^{-1}, \quad Q = DG^{-1}\kappa(1+\kappa)^{-1}$$

$$f(p) = 1/2 Q\kappa^{-1}(p+1)(p+2) \{ [Na^p \cos(p-1)\alpha - Sb^p \sin(p-1)\alpha](\Delta_2^{-1})^{-1} + Na^p \sin(p-1)\alpha + Sb^p \cos(p-1)\alpha \}$$

Here the contour L_1 is the line $\text{Re } p = \lambda - 3$. If the integrand in the second integral in (1.15) satisfies conditions (1) and (2) in the strip $\lambda - 3 \leq \text{Re } p \leq \lambda$, then the contour L_1 can again be shifted and the following difference equation can be obtained on L

$$B(p+3) = F_1(p)F_2(p)B(p) + f(p) \tag{1.16}$$

Finding the canonical solution $B_0(p)$ of the homogeneous equation (1.16) with coefficient $F_1(p)$ by the Barnes method [6], with the coefficient $F_2(p)$ by the method of Bantsuri [5], we obtain

$$B_0(p) = Q^{1/3} \sin \frac{\pi p}{6} \Gamma(p) X(p) \tag{1.17}$$

$$X(p) = K^{-1}(p) Y(p) \quad (-3 < \text{Re } p \leq 0), \quad X(p) = Y(p) \quad (0 < \text{Re } p \leq 3)$$

$$Y(p) = \exp \left\{ \frac{1}{6i} \int_{-i\infty}^{i\infty} \text{ctg} \frac{\pi(t-p)}{3} \ln K(t) dt \right\}$$

$$K(p) = F_2(p) \text{tg} \frac{\pi p}{6}$$

The canonical solution $B_{10}(p)$ of the homogeneous equation

$$B_1(p+3) = -F_1(p)F_2(p)B_1(p)$$

is expressed by the formula $B_{10}(p) = B_0(p) \cos(1/3 \pi p)$. Taking into account the constraints (1), (2) and the beam equilibrium conditions (1.3), the general solution of the inhomogeneous equation (1.16) is written as

$$B(p) = B_0(p) \left(C_1 \text{ctg} \frac{\pi p}{3} + C_2 \right) + B_{10}(p) Z(p) \text{ctg} \frac{\pi p}{3} \tag{1.18}$$

Substituting (1.18) into (1.16), we obtain

$$Z(p+3) = -Z(p) - g(p), \quad g(p) = -B_{10}^{-1}(p+3) f(p) \text{tg} \frac{\pi p}{3} \tag{1.19}$$

Analogously to the solution (1.13)

$$Z(p) = W(p) - g(p), \quad (-3 < \text{Re } p \leq 0), \tag{1.20}$$

$$Z(p) = W(p) \quad (0 < \text{Re } p \leq 3)$$

$$W(p) = \frac{1}{6i} \int_{-i\infty}^{i\infty} \frac{g(t) dt}{\sin 1/3 \pi (t-p)}$$

The integral conditions (1.3) whose left sides are the transforms of the function

$\sigma_\theta(r, 0)$ at the points $p = 0$ and $p = 1$, yield the following two equations to evaluate the constants C_1 and C_2 :

$$\begin{aligned} \frac{1}{2} (1 + \kappa) X(0) [C_1 + Z(0)] &= -P \\ \frac{1}{2} (1 + \kappa) Q^{1/2} Y(1) \left[\frac{C_1}{\sqrt{3}} + C_2 + \frac{W(1)}{2\sqrt{3}} \right] &= Pl - M \end{aligned}$$

Verifying conditions (1) and (2) causes no difficulties here because of the exponential decrease in the functions $g(p)$ and $B(p)$ as $|\text{Im } p| \rightarrow \infty$.

The nature of the stresses $\sigma_\theta(r, 0)$ at the angular point of the wedge is determined by the residue of the function $(1 + \kappa)pr^{-p-1}B(p)$ at the first pole to the left of L . Since the zeros of the functions $F_2(p)$ and $F(p)$ coincide, $\sigma_\theta(r, 0) = O(r^{\alpha-1})$ for $\alpha > \alpha^*$ as in problem (a).

2. We consider a more complex version of the load applied to the beam in the contact zone. Let the elastic wedge $0 \leq \theta \leq \alpha$ be reinforced by a beam slab of constant thickness h on the $\theta = 0$ side (Fig. 2).

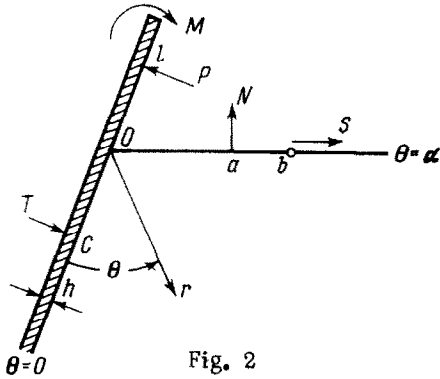


Fig. 2

There is no friction between the beam and the wedge, the free parts of the wedge $\theta = \alpha$ and the beam $\theta = \pi$ are loaded as in problem (b), and the normal force T acts on the beam at the point $r = c, \theta = 0$.

This problem is determined by conditions (1.2) and the conditions

$$\begin{aligned} \tau_{r\theta}(r, 0) &= 0, \quad D_1 \frac{\partial^4 v(r, 0)}{\partial r^4} = \quad (2.1) \\ \sigma_\theta(r, 0) &= T\delta(r - c), \quad D_1 = \frac{Eh^3}{12(1 - \mu^2)} \end{aligned}$$

$$\int_0^\infty \sigma_\theta(r, 0) dr = P - T, \quad \int_0^\infty \sigma_\theta(r, 0) r dr = M - Pl - Tc \quad (2.2)$$

Let us seek its solution in the form (1.5) where by virtue of (2.1)

$$B(p) = (\Delta_2^+ p)^{-1} [A(p)p(\Delta_1^+ - p - 1) - Na^p \sin(p - 1)\alpha + S b^p \cos(p - 1)\alpha]$$

the function $A_1(p)$ should satisfy conditions (1) and (2) in the strip $\lambda \leq \text{Re } p \leq \lambda + 3$ and the difference equation

$$A_1(p + 3) = F_1(p)F_2(p)A_1(p) + f(p) \quad (2.3)$$

on the contour L . Here

$$A_1(p) = 2F_2^{-1}(p)[pA(p) - f(p)], \quad F_1(p) = Q(p + 1)(p + 2)(p + 3) \quad (2.4)$$

$$Q = \frac{1}{4} D_1 G^{-1}(1 + \kappa), \quad F_2(p) = \frac{1}{2} \Delta_2^+ \Delta_4^{-1}, \quad \Delta_4 = \sin^2 p\alpha - p^2 \sin^2 \alpha$$

$$f(p) = \frac{1}{4} \Delta_4^{-1} \{ (\Delta_1^+ + p - 1) [Na^p \sin(p - 1)\alpha - S b^p \cos(p - 1)\alpha] - \Delta_2^+ [Na^p \cos(p - 1)\alpha - S b^p \sin(p - 1)\alpha] - T c^{p+3} \}$$

Just as (1.8), the solution of the problem (2.3) is

$$A_1(p) = A_{10}(p) \left[C_1 \operatorname{ctg} \frac{\pi p}{3} + C_2 + Z(p) \cos \frac{\pi p}{3} \operatorname{ctg} \frac{\pi p}{3} \right] \quad (2.5)$$

$$A_{10}(p) = Q^{1/3} \sin \frac{\pi p}{6} \Gamma(p+1) X(p)$$

Here the function $X(p)$ is expressed by the formulas (1.17), $F_2(p)$ and Q by (2.4), the function $Z(p)$ has the form (1.20), where

$$g(p) = \operatorname{tg} (1/3 \pi p) f(p) [A_{10}(p+3) \cos (1/3 \pi p)]^{-1} \quad (2.6)$$

From the conditions (2.2) we obtain

$$C_1 = \frac{2(P-T)-Z(0)}{X(0)}, \quad C_2 = \frac{2(M-Pl-Te)}{Q^{1/3} Y(1)} - \frac{C_1}{\sqrt{3}} - \frac{W(1)}{2\sqrt{3}}$$

The presence of a load on the beam makes the function $f(p)$ nondecreasing at infinity in the strip $\lambda \leq \operatorname{Re} p \leq \lambda + 3$ (due to the component Tc^{p+3}) and requires a stricter estimate of the function $A_1(p)$ for the verification of condition (2). If $\operatorname{Re} p > 0$, then according to (2.5) and (1.20), the function $Z(p)$ and $A_1(p)$ as well decrease exponentially when $|\operatorname{Im} p| \rightarrow \infty$. If $\operatorname{Re} p \leq 0$, then the result required follows from (2.4)–(2.6) and (1.20)

$$\begin{aligned} |A_1(i\beta + \lambda_1)| &= |A_{10}(i\beta + \lambda_1) A_{10}^{-1}(i\beta + \lambda_1 + 3)| O(1) = \\ &O(|\beta|^{-3}) \quad (\lambda_1 \leq 0) \end{aligned}$$

In this problem $\alpha^* = 1/2 \pi$. For $\alpha < \alpha^*$ we have $\sigma_0(0, 0) = O(1)$, for $\alpha > \alpha^*$ and $r \rightarrow 0$ we have $\sigma_0(r, 0) = O(r^{a_1-1})$, where a_1 is the first positive zero of the function Δ_2^+ . If $\alpha = \pi, 3/2 \pi, 2\pi$, then $a_1 = 1/2, 1/3, 1/4$, respectively.

For a half-plane, the considered problem is solved for $\alpha = \pi$ by another method in [7].

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